The heating source was a continuous mode IAG laser, producing power levels up to 80 W. The beam diameter at the exit window was 4 mm. The experimental vessel was made of glass, with dimensions of 60 × 60 × 200 mm.

During the experiments temperature distribution profiles were determined at various sections of the induced convective flow. The vertical temperature profile for the liquid PES-4 is shown in Fig. 1. The liquid column height was 180 mm, with heating power of 45 W, radiation propagating vertically downward.

A study was also made of the dependence of temperature on the jet axis on heating radiation power (Fig. 2). It was found that for all liquids these curves can be approximated well by the expression

$$\Delta T(0) = AP^{0.36},$$

where A depends on the height of the column and physical properties of the liquid.

NOTATION

ε, refraction angle; p, impact parameter; n, index of refraction; n_0, index of refraction of undisturbed medium; $\Delta T(r)$, temperature change at measurement point; dn/dT, temperature dependence of index of refraction; ν, kinematic viscosity; C_p, specific heat; λ, thermal conductivity; P, heating radiation power; A, experimental constant.

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NONLINEAR WATER TRANSPORT IN INTRASOIL IRRIGATION

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Solutions have been obtained for nonlinear capillary diffusion from instantaneous planar, line, and point sources with allowance for plant root uptake.

New irrigation methods have various advantages over traditional spray and trench ones, as a system of perforated tubes or droplet feeds is buried at a certain depth, which reduces the water loss by evaporation by comparison with surface supply. Such buried irrigation can be optimized if various aspects of hydrodynamics and transport theory can be resolved for unsaturated soil with allowance for uptake by the roots.

General principles and methods have been given in heat and water transport theory for porous media in [1-3]. Even under isothermal conditions, the transport in an incompletely saturated medium involves extremely complicated nonlinear boundary-value problems. These difficulties have led to various linearization techniques being used [4-6] or various numerical methods [7-9]. Many detailed aspects are thereby neglected. Here we consider these boundary-value problems containing marked nonlinearity for planar, axial, and central symmetries without resort to linearization, although we neglect the bound water (and thus the space excluded from the transport) and the nonzero retained water content (at which the plants

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cease to take up water from the soil), while we also assume that the gravitational component in the total head is small by comparison with the capillary one (negative capillary pressure).

With these assumptions, the central equation in Klute's theory [3, 10] becomes

$$\partial w/\partial t = \nabla (k\nabla\psi) - q(w). \tag{1}$$

If the saturation is incomplete, the approximate relation $w = w_0 \exp(a\psi)$ applies between the water content and the capillary pressure [2, 11], and the approximation $k = k_0 w^n$ applies for the water transport coefficient, where the values suggested for n are 3 [2], 3.56 [12], and 4 [13]. Therefore, one can take $d\psi/dw = 1/aw$ and

$$\nabla (k\nabla\psi) = D\nabla (w^{n-1}\nabla w), \quad D = k_0/a. \tag{2}$$

The uptake rate per unit volume can be represented to a first approximation [13] as

$$q(w) \doteq \varkappa w; \tag{3}$$

however, more complicated methods of describing the uptake have been proposed [7, 14, 15].

From (2) and (3), (1) is written as

$$\partial w/\partial t = D\nabla (w^{n-1}\nabla w) - \varkappa w. \tag{4}$$

(2) and (3), as well as (4), apply only for incomplete saturation, when the water content is less than a critical value. In the propagation considered below from instantaneous sources, we assume that the initial water content $w(0, x)$ is zero and that the water content $w(t, x)$ away from the sources becomes zero. We also assume as given the total amount of water entering the soil from the source at the start.

We transform (4) by means of a new unknown function and a time variable:

$$w(t, x) = \exp(-\varkappa t) f(\tau, x), \quad \tau = \frac{D}{n(n-1)\varkappa}\{1 - \exp[-(n-1)\varkappa t]\}. \tag{5}$$

Then small sources with planar, cylindrical, and spherical symmetries give the boundary-value treatments for $f(\tau, x)$ as

$$\begin{gathered}\frac{\partial f_s}{\partial \tau} = \frac{1}{x^s}\frac{\partial}{\partial x}\left(x^s \frac{\partial f_s^n}{\partial x}\right); \quad \int_0^\infty x^s f_s(0, x)\,dx = Q; \\ \partial f_0/\partial x = 0,\ x = 0; \quad f_s(\tau, \infty) = 0; \quad f_s(0, x) = 0,\end{gathered} \tag{6}$$

where parameter s is correspondingly 0, 1, or 2. The condition at $x = 0$ in the planar case follows from the symmetry.

Equations (6) have self-similar solutions, which were first derived in [16, 17] in the propagation of strong thermal waves and nonlinear filtration [18]. For planar, axisymmetric and spherically symmetrical cases, one introduces the self-similar variable and the new unknown function $\phi(\xi)$ [17]:

$$\begin{gathered}s = 0: \xi = x(Q^{n-1}\tau)^{-\frac{1}{n+1}}, \quad f_0 = (Q^2/\tau)^{\frac{1}{n+1}}\varphi_0(\xi); \\ s = 1: \xi = x(Q^{n-1}\tau)^{-\frac{1}{2n}}, \quad f_1 = (Q/\tau)^{\frac{1}{n}}\varphi_1(\xi); \\ s = 2: \xi = x(Q^{n-1}\tau)^{-\frac{1}{3n-1}}, \quad f_2 = (Q^{\frac{2}{3}}/\tau)^{\frac{3}{3n-1}}\varphi_2(\xi).\end{gathered} \tag{7}$$

From (6) one gets ordinary differential equations and conditions for them for $\phi_s(\xi)$; the integral condition in (6) applies not only for $\tau = 0$ but also for any $\tau > 0$, and correspondingly the $\phi_s(\xi)$ satisfy an integral condition containing $Q = 1$ similar in form. The solutions are [17]

$$\varphi_s(\xi) = \begin{cases} C_s\left[1 - \left(\dfrac{\xi}{\xi_s}\right)^2\right]^{\frac{1}{n-1}}, & 0 \leqslant \xi \leqslant \xi_s, \\ 0, & \xi \geqslant \xi_s, \end{cases} \tag{8}$$

where for a planar source

$$\xi_0=\left\{2\left[\frac{2n(n+1)}{n-1}\right]^{\frac{1}{n-1}}\Gamma\left(\frac{1}{2}+\frac{n}{n-1}\right)\left[\Gamma\left(\frac{n}{n-1}\right)\Gamma\left(\frac{1}{2}\right)\right]^{-1}\right\}^{\frac{n-1}{n+1}},\qquad(9)$$

$$C_0=\left[\frac{n-1}{2n(n+1)}\right]^{\frac{1}{n-1}}\xi_0^2,$$

for a line source

$$\xi_1=\left\{2\left(\frac{4n^2}{n-1}\right)^{\frac{1}{n-1}}\frac{n}{n-1}\right\}^{\frac{n-1}{2n}},\quad C_1=\left(\frac{n-1}{4n^2}\right)^{\frac{1}{n-1}}\xi_1^2\qquad(10)$$

and for a point source

$$\xi_2=\left\{2\left[\frac{2n(3n-1)}{n-1}\right]^{\frac{1}{n-1}}\Gamma\left(\frac{3}{2}+\frac{n}{n-1}\right)\times\right.$$

$$\left.\times\left[\Gamma\left(\frac{n}{n-1}\right)\Gamma\left(\frac{3}{2}\right)\right]^{-1}\right\}^{\frac{n-1}{3n-1}},\quad C_2=\left[\frac{n-1}{2n(3n-1)}\right]^{\frac{1}{n-1}}\xi_2^2.\qquad(11)$$

Here $\Gamma(z)$ is Euler's gamma function. The characteristics of (8)-(11) have been discussed in detail in [16-18], as they describe a wave whose front x_* is

$$s=0:x_*=\xi_0(Q^{n-1}\tau)^{\frac{1}{n+1}};$$
$$s=1:x_*=\xi_1(Q^{n-1}\tau)^{\frac{1}{2n}};\qquad(12)$$
$$s=2:x_*=\xi_2(Q^{n-1}\tau)^{\frac{1}{3n-1}}$$

and propagates with a finite velocity decreasing monotonically as τ increases. Here $f_s(\tau, x)$ is dependent only weakly on x at $(0, x_*)$, apart from a small range directly adjoining the front; the values for a given ξ/ξ_s decrease in inverse proportion to fractional powers of τ as indicated in (7). For $n < 2$, the $f_s(\tau, x)$ curves for a given τ approach the x axis and touch it at x_*, while for $n > 2$, they do so at a right angle [17]. The front recedes asymptotically ($\tau \to \infty$) from the source to an infinite distance. The functions $\phi_s(\xi)$ and $f_s(\tau, x)$ satisfy

$$\lim_{\xi\to 0}\xi^s\frac{d\varphi_s^n(\xi)}{d\xi}=0,\quad \lim_{x\to 0}x^s\frac{\partial f_s^n(\tau,\ x)}{\partial x}=0,\qquad(13)$$

which reflect the source being instantaneous.

These solutions from (5), (7), and (8) are written explicitly as: for a planar source

$$w(t,\ x)=\frac{W_0\exp(-\varkappa t)}{\{1-\exp[-(n-1)\varkappa t]\}^{\frac{1}{n+1}}}\left[1-\left(\frac{\xi}{\xi_0}\right)^2\right]^{\frac{1}{n-1}},$$

$$\xi(t,\ x)=\left[\frac{n(n-1)\varkappa}{DQ^{n-1}}\right]^{\frac{1}{n+1}}\frac{x}{\{1-\exp[-(n-1)\varkappa t]\}^{\frac{1}{n+1}}},\qquad(14)$$

$$W_0=C_0\left[\frac{n(n-1)\varkappa Q^2}{D}\right]^{\frac{1}{n+1}},$$

for a line source

$$w(t, x) = \frac{W_1 \exp(-\varkappa t)}{\{1 - \exp[-(n-1)\varkappa t]\}^{\frac{1}{n}}} \left[1 - \left(\frac{\xi}{\xi_1}\right)^2\right]^{\frac{1}{n-1}},$$

$$\xi(t, x) = \left[\frac{n(n-1)\varkappa}{DQ^{n-1}}\right]^{\frac{1}{2n}} \frac{x}{\{1 - \exp[-(n-1)\varkappa t]\}^{\frac{1}{2n}}}, \quad (15)$$

$$W_1 = C_1 \left[\frac{n(n-1)\varkappa Q}{D}\right]^{\frac{1}{n}},$$

and for a point source

$$w(t, x) = \frac{W_2 \exp(-\varkappa t)}{\{1 - \exp[-(n-1)\varkappa t]\}^{\frac{3}{3n-1}}} \left[1 - \left(\frac{\xi}{\xi_2}\right)^2\right]^{\frac{1}{n-1}},$$

$$\xi(t, x) = \left[\frac{n(n-1)\varkappa}{DQ^{n-1}}\right]^{\frac{1}{3n-1}} \frac{x}{\{1 - \exp[-(n-1)\varkappa t]\}^{\frac{1}{3n-1}}}, \quad (16)$$

$$W_2 = C_2 \left[\frac{n(n-1)\varkappa Q^{2/3}}{D}\right]^{\frac{3}{3n-1}},$$

where the constants ξ_s and C_s are defined correspondingly in (9), (10), and (11). Solutions (14)-(16) are not self-similar. As κ tends to zero (i.e., when the roots cease to take up water), these solutions become the self-similar (8) ones.

The most marked difference between propagation with absorption from that without it is that even asymptotically for $t \to \infty$, the front is displaced only to a maximum but finite distance x_m from the source. (12) gives x_m on the basis of τ represented in terms of t from (5) as

$$s = 0: x_m = \xi_0 \left[\frac{Q^{n-1}D}{n(n-1)\varkappa}\right]^{\frac{1}{n+1}};$$

$$s = 1: x_m = \xi_1 \left[\frac{Q^{n-1}D}{n(n-1)\varkappa}\right]^{\frac{1}{2n}}; \quad (17)$$

$$s = 2: x_m = \xi_2 \left[\frac{Q^{n-1}D}{n(n-1)\varkappa}\right]^{\frac{1}{3n-1}},$$

i.e., x_m increases with Q and D but decreases as κ increases. Then with uptake, any instantaneous source with a given output can irrigate only a quite definite finite volume. The falls in front propagation rate and water content within the irrigated zone follow almost exponential laws as t increases.

The finite irrigated volume imposes constraints on the typical distances between adjacent pipes or holes, as well as between perforated tubes and planar networks of such tubes. If the object is to produce more or less uniform irrigation for an entire given soil volume, the distances should in any case be less than twice the corresponding x_m in (17). When one is choosing appropriate distances between adjacent sources, importance attaches to the total amount of water taken up by the roots at a given distance from the source. The general expression for this is

$$G(x) = \varkappa \int_{T(x)}^{\infty} w(t, x)\, dt = \int_{0}^{z(x)} w(-\varkappa^{-1} \ln \zeta, x) \frac{d\zeta}{\zeta}, \quad (18)$$

$$\zeta = \exp(-\varkappa t), \quad \xi(T, x) = \xi_s, \quad z(x) = \exp(-\varkappa T),$$

where w(t, x) and ξ(t, x) are defined by (14)-(16). Similarly, one can represent the amount of water taken up in a finite time (e.g., between successive irrigations). The integrals in (18) as functions of s cannot be expressed in terms of elementary functions.

The integrals in (18) can be represented as incomplete beta functions or Gauss hypergeometric ones, but this is of no particular value as one lacks fairly detailed tables for these functions. We therefore derive a simple approximation for G(x) for large t (when $\zeta^{n-1} = \exp[-(n-1)\kappa t] \ll 1$) or, which is the same, for x sufficiently close to x_m (i.e., in the region particularly hazardous as regards insufficient water supply). That approximation also simplifies the w(t, x) expression considerably; from (14)-(17) we have

$$w(t, x) \approx W_s\left[1 - \left(\frac{x}{x_m}\right)^2\right]^{\frac{1}{n-1}} \exp(-\varkappa t),$$

where (18) implies

$$z(x) = \left[1 - \left(\frac{x}{x_m}\right)^\alpha\right]^{\frac{1}{n-1}},$$

in which α = n + 1, 2n, and 3n − 1 for s = 0, 1, and 2 correspondingly. Finally, (18) gives

$$G(x) \approx W_s\left\{\left[1 - \left(\frac{x}{x_m}\right)^2\right]\left[1 - \left(\frac{x}{x_m}\right)^\alpha\right]\right\}^{\frac{1}{n-1}}. \qquad (19)$$

This formula also applies for the amount of water taken up from a single irrigation if the intervals between irrigations are long enough.

If agrotechnical considerations impose specifications for the absorbed-water distribution within the irrigated volume, (19) is useful to locate the sources properly and define the necessary distances between them. As G(x) is dependent inexplicitly on Q via x_m from (17), that formula can also be used in irrigation tactics, e.g., in determining the optimum number and extent of the irrigation cycles.

One can use the results for the different source types simultaneously in applications. For example, one might have a parallel set of perforated tubes, where the point source result can be used to simulate individual holes and thus determine the optimum distance between those holes. At the next scale level, when the waves have joined up from the holes, each tube can be represented as a line source, whose output is determined by averaging along it. Then one can define the general tube arrangement (linear, chessboard, and so on) and the distances between tubes. Finally, a system of tubes in one plane can be represented as a planar source, whose output is governed by averaging over all tubes. The solution for a planar source can also be used to describe surface irrigation if evaporation can be neglected. Similarly, a line source can be useful in simulating trench surface irrigation.

Allowance for the finite dimensions means that the solutions to (6) with different s cease to be self-similar, so (14)-(16) are approximate but describe the propagation adequately at distances considerably exceeding the source dimensions. The same applies to the assumption about the instantaneous sources. The formulas apply approximately for times substantially exceeding the characteristic working times.

The equations in (6) have comparatively simple self-similar solutions also if $f_0(\tau, x)$ or the derivatives $\partial f_s^n(\tau, x)/\partial x$ follow power laws at x = 0 with any positive indices [17]; these correspond to the solutions for (4) of this type in which the source output is a one-parameter time function, whose form is readily derived from (5) and the power law. For a planar source, there is also a solution corresponding to the water-content change at x = 0 on a one-parameter law. Although of themselves these solutions correspond to somewhat hypothetical situations, they are useful for solving many nonlinear boundary-value problems approximately for the propagation of water based on integral relationships, by analogy with boundary-layer theory. In [19], there is a discussion of this in relation to certain one-dimensional treatments in nonlinear filtration theory.

When this paper had been sent to press, the authors became aware of [20], in which solutions were considered to a quasilinear thermal-conduction equation analogous to those discussed here.

NOTATION

C_s, constants introduced in (8); D, diffusivity; f, unknown function introduced in (5); G, total water uptake in unit volume; k, water transport coefficient; Q, quantity proportional to source output; q, root uptake in unit volume in unit time; s, parameter in (6); T, integration limit in (18); t, time; W_s, constants defined in (14)-(16); w, water content; x, spatial coordinate; x_*, coordinate of water-content front; x_m, maximum distance traveled by front; z, integration limit in (18); α, exponent in (19); ζ, variable in (18); κ, absorption coefficient; ξ, self-similar variable; ξ_s, constants introduced in (8); τ, modified time; ψ, negative capillary pressure.

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